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B.Sc Sem-4 MIC

Que (1) :- Derive and Prove The integral Test.

Theorem :- If $F(x) > 0$, when $x > 0$ and if $F(x)$ decreases as x increases then the sequences

$$S_n = F(1) + F(2) + F(3) + \cdots + F(n)$$

$$I_n = \int_1^n F(x).dx$$

are either both convergent or both divergent

Proof: Since $F(x) > 0$ when $x > 0$ the sequence S_n and the sequence (I_n) both are monotonic increasing

Also Since $F(x)$ is a monotonic decreasing function of x. therefore when

$$n \leq x \leq n+1$$

$$F(n) \geq F(x) \geq F(n+1) \quad \text{——— (1)}$$

The equality (=) sign covers the case when $F(x)$ remains Constant from $x = n$ to $x = n+1$

Now integrating (1) with respect to x from the limit $x = n$ to $x = n+1$, we have.

$$\int_n^{n+1} F(n)\,dx \geq \int_n^{n+1} F(x).dx \geq \int_n^{n+1} F(n+1).dx$$

$$\Rightarrow F(n)\,[x]_n^{n+1} \geq \int_n^{n+1} F(x).dx \geq F(n+1)\,[x]_n^{n+1}$$

$$\Rightarrow F(n) \geq \int_n^{n+1} F(x).dx \geq F(n+1)$$

$$\Rightarrow F(n) \geq \int_1^{n+1} F(x).dx - \int_1^{n} F(x).dx \geq F(n+1)$$

$$\Rightarrow F(n) \geq I_{n+1} - I_n \geq F(n+1) \quad \text{——— (2)}$$

Putting $n = 1, 2, 3, \ldots (n-1)$ in (2) and nothing that

$$I_1 = \int_1^1 F(x)\,dx = 0$$, we get

$$F(1) \geq l_2 - l_1 \geq F(2)$$
$$F(2) \geq l_3 - l_2 \geq F(3)$$
$$F(3) \geq l_4 - l_3 \geq F(4)$$
$$\cdots\cdots\cdots$$
$$F(n-1) \geq l_n - l_{n-1} \geq F(n)$$

Adding these. We get

$$S_n - F(n) \geq l_n \geq S_n - F(1)$$

i.e $$S_n - F(1) \leq l_n \leq S_n - F(n) \quad \text{——} \quad (3)$$

Now suppose that l_n tends to a finite limit l as $n \to \infty$
Since (l_n) is monotonic increasing we have $l_n \leq l$ and
so from the first and second inequalities of (3)

$$S_n \leq l_n + F(1) \quad \text{——} \quad (4)$$

i.e $$S_n \leq l + F(1)$$

But $l + F(1)$ is independent of n. so that by (4)
(S_n) is a monotonic increasing sequence which is bounded above the upper bound being $l + F(1)$

Hence $S_n \to$ a finite limit S such that

$$S \leq l + F(1)$$

Next suppose that $l_n \to \infty$
Then from the second and third inequalities of (3)

$$S_n \geq l_n + F(n)$$

and if $l_n \to \infty$ then $S_n \to \infty$

Similarly if $S_n \to$ a finite limit S (3) gives

$$l_n < S_n - F(n) \leq S_n \leq S$$

and therefore $l_n \to l \leq S$

Also if $S_n \to \infty$. then since $l_n > S_n$

$$-F(1) \cdot l_n \to \infty$$

Hence the theorem